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Partitions of the reals and models of ZF

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Abstract

We consider several partition relations and describe models of ZF which can be used to distinguish between them. This is an extended abstract of a talk delivered in the RIMS Symposium on Axiomatic Set Theory and Set Theoretic Topology, held at RIMS University of Kyoto, 28-30 November 2008.

1 Introduction.

We consider partitions of the Baire space ω^ω of all infinite sequences of natural numbers with the product topology obtained giving to ω the discrete topolgy, and also partitions of its closed subspace $[\omega]^\omega$ of all infinite subsets of ω, which can be identified with the strictly increasing sequences of natural numbers. If A is an infinite set of natural numbers, we use $[A]^\omega$ to denote the set of infinite subsets of A.

Definition 1 *Given $n \in \omega$, we say that a partition $c : [\omega]^\omega \to n$ is Ramsey if there is $H \in [\omega]^\omega$ such that c is constant on $[H]^\omega$. Such a set H is said to be homogeneous for c.*

One of the emblematic results in this area is the following theorem of F. Galvin and K. Prikry

Theorem 2 *[5] For every $n \in \omega$, every Borel measurable partition $c : [\omega]^\omega \to n$ is Ramsey.*

The notation

$$\omega \underset{\Gamma}{\to} (\omega)^\omega_n$$

is used to express that for every Γ-measurable $c : [\omega]^\omega \to n$, there is $H \in [\omega]^\omega$ such that c is constant on $[H]^\omega$. So, the Galvin-Prikry theorem is

$$\forall n\ (\omega \underset{\text{Borel}}{\longrightarrow} (\omega)^\omega_n).$$

If no class Γ is mentioned, the partition symbol refers to all functions $c : [\omega]^\omega \to n$. Also, if $n = 2$, the subindex is usually omitted.

It is well known that $\omega \to (\omega)^\omega$ implies that there are no non-principal ultrafilters on ω; so, ZFC proves that this partition relation is false. Nevertheless, a celebrated result of Mathias [7] shows that this partition relation is consistent with $ZF + DC$, provided that the existence of an inaccessible cardinal is consistent.

2 Infinite partitions.

It is easy to find a clopen non-Ramsey partition of $[\omega]^\omega$ into infinitely many pieces. Namely, $h : [\omega]^\omega \to \omega$ defined by $h(A) = \min(A)$. Thus, ZF proves $\omega \not\to (\omega)^\omega_\omega$

It is interesting to consider a version of $\omega \to (\omega)^\omega_\omega$ that requires only the existence of a set of the form $[H]^\omega$ which avoids a piece of the partition, instead of requiring that it is contained in a single piece. For this type of partition relation it is customary to use the following notation. The expression

$$\omega \underset{\Gamma}{\to} [\omega]^\omega_K$$

means that for every Γ-measurable $c : [\omega]^\omega \to K$, there is $H \in [\omega]^\omega$ such that $c"[H]^\omega \subsetneq K$.

It is straightforward to verify that this partition relation holds for Borel partitions, but again, the Axiom of Choice implies that there are partitions of $[\omega]^\omega$ into infinitely many pieces for which every set of the form $[H]^\omega$ meets every piece. In fact, we have the following.

Proposition 3 *If there is a non-principal ultrafilter on ω, then*

$$\omega \not\to [\omega]^\omega_{2^\omega}.$$

Actually, a weaker hypothesis is enough to refute the partition relation

$$\omega \to [\omega]^\omega_{2^\omega},$$

namely, the existence of a non-principal non-meager filter on ω. This result is part of ongoing work done jointly with S. Todorcevic and will appear elsewhere.

3 Homogeneous sublattices and perfect sets.

We now turn to a different type of partition property, which was first considered in [4].

We use the symbol

$$\omega \underset{\Gamma}{\to} ((\omega))^\omega_n$$

to express that for every Γ-measurable function $c : [\omega]^\omega \to n$, there are $A, B \in [\omega]^\omega$, wtih $A \subseteq B$ and $B \setminus A \in [\omega]^\omega$, such that c is constant on the sublattice of subsets of B given by $[A, B] = \{X \subseteq B : A \subseteq X\}$.

It is easily seen that the relation

$$\omega \xrightarrow[\text{Borel}]{} ((\omega))^\omega_n$$

follows from

$$\omega \xrightarrow[\text{Borel}]{} (\omega)^\omega_n.$$

And just as in the case of $\omega \to (\omega)^\omega$, the existence of a non-principal ultrafilter on ω implies that $\omega \not\to ((\omega))^\omega$.

The third type of partition relation we consider here is denoted by

$$\omega \xrightarrow[\Gamma]{} (\text{perfect})^\omega_n$$

meaning that for every Γ-measurable function $c : [\omega]^\omega \to n$, there is a perfect set $P \subseteq [\omega]^\omega$ on which c is constant.

A Bernstein set is just a counterexample to $\omega \to (\text{perfect})^\omega$, this is, a set B with the property that both B and its complement meet every perfect set. Such a set can be obtained from a well ordering of the reals.

In his article [8] Solovay, assuming the consistency of inaccesible cardinals, constructed a model of ZF where every set of reals is Lebesgue measurable, has the property of Baire, and if not countable, contains a perfect subset. Of course, the axiom of choice does not hold in this model, although the axiom of dependent choices does. In general, a model M of ZF is said to be a Solovay model if it is (elementary equivalent to) the model $L(\mathbb{R})$ computed in the Levy collapse of an inaccessible cardinal to $\aleph_1$. The result of Mathias mentioned above ([7]), establishes that the partition property $\omega \to (\omega)^\omega$ holds in Solovay models; therefore the same is true for the properties $\omega \to ((\omega))^\omega$, $\omega \to [\omega]^\omega_{2^\omega}$, and $\omega \to (\text{ perfect })^\omega$ which follow from it.

Consider now the model $L(\mathbb{R})[\mathcal{U}]$ obtained adding a selective ultrafilter to a Solovay model $L(\mathbb{R})$ using the poset of infinite subsets of ω ordered by the relation of almost containment.

It was shown in [2] that $\omega \to (\text{ perfect })^\omega$ holds in $L(\mathbb{R})[\mathcal{U}]$. This was done proving that in Solovay models, the following parameterized partition relation holds: for every $n \in \omega$ and every $c : [\omega]^\omega \times \omega^\omega \to n$, there is $H \in [\omega]^\omega$ and a perfect set $P \subseteq \omega^\omega$ such that c is constant on the product $[H]^\omega \times P$.

Therefore, the existence of a non-principal ultrafilter on ω is a consequence of the Axiom of Choice not strong enough to produce a Bernstein set. By our previous remarks about non-principal ultrafilters, none of the other properties hold in the model $L(\mathbb{R})[\mathcal{U}]$, since in it $\mathcal{U}$ is non-principal ultrafilter on ω.

4 Cohen extensions

Adding Cohen generic reals to the constructuble universe L, we obtain a model in which

$$\omega \underset{\text{Projective}}{\longrightarrow} ((\omega))^{\omega}$$

holds but there is a $\mathbf{\Delta}^1_2$ counterexample for $\omega \to (\omega)^{\omega}$.

We start from L, and add ω_1-many Cohen genric reals using the ω_1 product of Cohen forcing with finite support. In, [1] it is shown that in this extension the partition relation $\omega \to ((\omega))^{\omega}$ holds for projective partitions.

It follows from [6], 2.2, that in this model there is a $\mathbf{\Delta}^1_2$ counterexample for $\omega \to (\omega)^{\omega}$, i.e. there is a $\mathbf{\Delta}^1_2$ non-Ramsey set.

In fact, the relation $\omega \to ((\omega))^{\omega}$ holds in the extension for partitions definable with real parameters, and so, it also holds in the inner model $L(\mathbb{R})$ of all the sets in the extension that are constructible from reals. In this way we obtain a model in which $\omega \to ((\omega))^{\omega}$ holds but $\omega \to (\omega)^{\omega}$ does not.

The model obtained adding ω_2-many Cohen generic reals to L offers additional features. For example, in this model there is a non-meager non-principal filter on ω. Taking the appropriate inner model we obtain a model in which $\omega \to ((\omega))^{\omega}$ holds, but $\omega \to [\omega]^{\omega}_{2^{\omega}}$ fails.

5 Conclusion.

Sumarizing, we have that $\omega \to (\omega)^{\omega}$ implies both $\omega \to ((\omega))^{\omega}$ and $\omega \to [\omega]^{\omega}_{2^{\omega}}$, the first implication being strict.

Each of the properties $\omega \to ((\omega))^{\omega}_2$ and $\omega \to [\omega]^{\omega}_{2^{\omega}}$ imply $\omega \to (\text{ perfect })^{\omega}$, and both implications are strict. The partition relation $\omega \to [\omega]^{\omega}_{2^{\omega}}$ is not implied by $\omega \to ((\omega))^{\omega}$.

Question: What is the exact relationship between the propereties $\omega \to (\omega)^{\omega}$ and $\omega \to [\omega]^{\omega}_{2^{\omega}}$? (See [3]).

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